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FLAT ENVELOPES IN COMMUTATIVE RINGS

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ABSTRACT

We prove that if R is a commutative ring such that each localization at a prime ideal has finite weak global dimension then every R-module has a flat envelope if and only if R is coherent and has weak global dimension less than or equal to two.

Flat preenvelopes and flat envelopes were defined by Enochs [2] which characterized the coherence in terms of flat preenvelopes and proved that for a noetherian local domain, every module has a flat envelope if the ring has weak global dimension $\leqq 2$, remarking that it would be interesting to determine the rings for which every module has a flat envelope. In [5] Martinez showed that, over a commutative ring with weak global dimension $\leqq 2$, every module has a flat envelope if and only if R is a coherent ring, and, for a domain, the condition that each module has a flat envelope is equivalent to being a coherent ring with weak global dimension $\leqq 2$.

In this paper R will be an associative ring with identity and an R-module will be a right R-module unless otherwise stated. All the morphisms, even of left R-modules, will be always written as acting on the left. If M is an R-module, and I a set, $M^{(I)}$ will be the direct sum of copies of M indexed by I, and $M^* = \mathrm{Hom}_R(M, R)$ will denote the dual module of M, β_M being the canonical homomorphism from M to M^{**}.

If M is an R-module, a projective cover of M is an epimorphism $f: P \rightarrow M$, where P is a projective R-module, such that $\mathrm{Ker}\, f$ is a superfluous submodule of P. It is easy to see that these conditions are equivalent to the existence of a homomorphism $f: P \rightarrow M$ with P projective verifying:

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(a) For each homomorphism $g: P' \to M$ where P' is a projective R-module, there exists a homomorphism $h: P' \to P$ such that $f \circ h = g$.

(b) Every endomorphism h of P such that $f \circ h = f$ is an automorphism.

Dually, a *flat preenvelope* of an R-module M is a homomorphism $f: M \to F$, with F a flat R-module, such that if g is a homomorphism from M to a flat R-module F' then there exists $h: F \to F'$ verifying $h \circ f = g$. If, furthermore, the endomorphisms h of F, satisfying $h \circ f = f$ are automorphisms, then f is called a *flat envelope* of M [2].

PROPOSITION 1. *Let R be a left coherent ring and M a finitely presented right R-module. Then M has a flat envelope if and only if the left R-module M^* has a projective cover.*

PROOF. Let $f: M \to F$ be a flat envelope of M. Then, by [2, Lemma 5.1], F is a finitely generated projective R-module. Let $f^*: F^* \to M^*$ be the induced homomorphism. We claim that f^* is a projective cover of M^*. To see this it will be enough to show that if $h: F^* \to F^*$ is a homomorphism such that $f^* \circ h = f^*$, then h is an automorphism (observe that f^* is an epimorphism because $f: M \to F$ is a flat envelope). Now, $\beta_F^{-1} \circ h^* \circ \beta_F$ is an endomorphism of F satisfying

$$\beta_F^{-1} \circ h^* \circ \beta_F \circ f = \beta_F^{-1} \circ h^* \circ f^{**} \circ \beta_M = \beta_F^{-1} \circ (f^* \circ h)^* \circ \beta_M$$

$$= \beta_F^{-1} \circ f^{**} \circ \beta_M = \beta_F^{-1} \circ \beta_F \circ f = f$$

so that $\beta_F^{-1} \circ h^* \circ \beta_F$ is an automorphism of F and hence $h = \beta_{F^*}^{-1} \circ h^{**} \circ \beta_{F^*}$ is an automorphism.

Conversely, if $g: F \to M^*$ is a projective cover of M^*, by the left coherence of R and the fact of M being a finitely presented right R-module, M^* is also finitely presented [1, Prop. 1] and hence F is a finitely generated projective module.

Now, let $f = g^* \circ \beta_M: M \to F^*$. In a similar way to the above case, if $h: F^* \to F^*$ is an homomorphism such that $h \circ f = f$, then h must be an automorphism.

On the other hand, let $t: M \to F'$ be a homomorphism where F' is a flat R-module. Then, as M is a finitely presented module, by [4, Theorem 1.2] t factors through an homomorphism $t': M \to P'$ where P' is a finitely generated projective R-module. Now $(P')^*$ is projective as left R-module and hence there exists a homomorphism $s: (P')^* \to F$ such that $g \circ s = (t')^*$, so that the homomorphism $\beta_{P'}^{-1} \circ s^*$ satisfies that $\beta_{P'}^{-1} \circ s^* \circ f = t'$ and we get that t factors through f, which completes the proof.

In [2] Enochs has shown that the left coherent rings are characterized by the condition that all the right R-modules have a flat preenvelope. More precisely we get the following result.

PROPOSITION 2. *Let R be a ring such that every finitely presented right R-module has a flat preenvelope. Then R is left coherent.*

PROOF. We will show that, for each set I, the right R-module R^I is flat. Let $f: M \to F$ be a flat preenvelope of a finitely presented right R-module M. Then the homomorphism from $\mathrm{Hom}_R(F, F')$ to $\mathrm{Hom}_R(M, F')$ induced by f is an epimorphism for every flat right R-module F'. In particular, this holds for R_R and hence, for each set I, $(f^*)^I : (F^*)^I \to (M^*)^I$ is an epimorphism. Now, since $(F^*)^I \cong \mathrm{Hom}_R(F, R^I)$ and $(M^*)^I \cong \mathrm{Hom}_R(M, R^I)$ we have that every homomorphism from M to R^I factors through a flat module, which shows that R^I is flat.

We recall that a ring R is called semiregular if each finitely presented right R-module (or equivalently each finitely presented left R-module) has a projective cover [6]. Von Neumann regular rings and semiperfect rings are examples of semiregular rings.

From Propositions 1 and 2 we can get

COROLLARY 3. *Let R be a semiregular ring. Then every finitely presented right R-module has a flat envelope if and only if R is a left coherent ring.*

COROLLARY 4. *Let R be a coherent commutative ring and $\mathfrak{p}$ a prime ideal of R. Then every finitely presented $R_{\mathfrak{p}}$-module has a flat envelope.*

A left R-module N is called FP-injective if $\mathrm{Ext}_R(M, N) = 0$ for each finitely presented left R-module M. R is said to be left self-FP-injective if itself considered as left R-module is FP-injective. We get

COROLLARY 5. *Let R be a right self-FP-injective right coherent ring. Then every finitely presented right R-module has a flat envelope if and only if R is a left coherent semiregular ring.*

PROOF. It follows from Corollary 3 and Proposition 1 because, in this case, every finitely presented left R-module is reflexive [3, Corol. 2.4].

Note that if R is a commutative ring such that every injective R-module is flat (an IF-ring) then every finitely presented R-module has a flat envelope if and only if R is a semiregular ring. This implies that there exist IF-rings without sufficient flat envelopes, even for finitely presented modules. For instance, let $R = \mathbf{Z} \ltimes \mathbf{Q}/\mathbf{Z}$ be the trivial extension of $\mathbf{Z}$ by $\mathbf{Q}/\mathbf{Z}$. By [1, Examp. 1, p. 249] R is a commutative IF-ring such that it is not regular modulo its radical and hence it is not a semiregular ring [6, Theorem 2.9].

In the remainder of the paper R will denote a commutative ring.

LEMMA 6. *Let M be an R-module with a flat preenvelope. Then $M_{\mathfrak{p}}$ has a flat preenvelope for each prime ideal $\mathfrak{p}$ of R.*

PROOF. Let $f : M \to F$ be a flat preenvelope of M and $\mathfrak{p}$ a prime ideal of R. If we denote by ϕ_M the canonical homomorphism from M to $M_{\mathfrak{p}}$, let $g : M_{\mathfrak{p}} \to F'$ be a homomorphism to a flat $R_{\mathfrak{p}}$-module F'. As F' is also a flat R-module there exists a homomorphism $h' : F \to F'$ such that $h' \circ f = g \circ \phi_M$. Then $h = h'_{\mathfrak{p}}$ verifies that

$$h \circ f_{\mathfrak{p}} \circ \phi_M = h \circ \phi_F \circ f = h' \circ f = g \circ \phi_M$$

and hence $h \circ f_{\mathfrak{p}} = g$ which implies that $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to F_{\mathfrak{p}}$ is a flat preenvelope of the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$.

In a straightforward way we can prove the following result.

LEMMA 7. *If M has a flat envelope $M \to F$ and $M \to P$ is a flat preenvelope of M, then there exists a flat envelope of M, $M \to F'$, such that F' is a direct summand of P and $F' \cong F$.*

PROPOSITION 8. *Let R be a ring for which every R-module has a flat envelope. Then for every finitely presented R-module M, the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}^{(\mathbf{N})}$ has a flat envelope for each prime ideal $\mathfrak{p}$ of R.*

PROOF. Let M be a finitely presented R-module and $f : M \to P$ a flat envelope of M. By hypothesis, the R-module $M^{(\mathbf{N})}$ has a flat envelope and as a consequence of [2, Coroll. 2 of Prop. 4.2] we get that if $\{g_n\}_{n \in \mathbf{N}}$ is a sequence of endomorphisms of P satisfying $g_n \circ f = 0 \ \forall n \in \mathbf{N}$, then for each $x \in P$ there exists $m_x \in \mathbf{N}$ such that $g_{m_x} \circ g_{m_x - 1} \circ \cdots \circ g_2 \circ g_1(x) = 0$.

Now, since P is a finitely generated R-module, it is possible to choose $m \in \mathbf{N}$ such that $g_m \circ \cdots \circ g_1 = 0$.

On the other hand, let $\{g'_n\}_{n \in \mathbf{N}}$ be a sequence of endomorphisms of $P_{\mathfrak{p}}$ with $g'_n \circ f_{\mathfrak{p}} = 0$ for each $n \in \mathbf{N}$. As P is a finitely generated projective R-module the canonical homomorphism π from $\mathrm{End}_R(P)_{\mathfrak{p}}$ to $\mathrm{End}_{R_{\mathfrak{p}}}(P_{\mathfrak{p}})$ is an isomorphism and hence for each $n \in \mathbf{N}$ there exist $g_n \in \mathrm{End}_R(P)$ and $s_n \in R \setminus \mathfrak{p}$ satisfying that $\pi(g_n/s_n) = g'_n$. In this way $\pi(g_n/1) = s_n \cdot g'_n$ and $s_n \cdot g'_n \circ f_{\mathfrak{p}} = 0$ for each $n \in \mathbf{N}$. If $\{x_1, \ldots, x_r\}$ is a generating set of M, we have that

$$(g_n \circ f)(x_i)/1 = (s_n \cdot g'_n \circ f_{\mathfrak{p}})(x_i/1) = 0 \qquad \text{for each } i \in \{1, \ldots, r\} \quad \text{and} \quad n \in \mathbf{N}.$$

So that there exist $t_n^i \in R \setminus \mathfrak{p}$ such that $(t_n^i \cdot g_n \circ f)(x_i) = t_n^i \cdot (g_n \circ f)(x_i) = 0$.

Now, if we consider $t_n = \Pi_{i=1}^r t_n^i$ we get that the sequence $\{t_n \cdot g_n\}$ satisfies that $t_n \cdot g_n \circ f = 0$ and hence there exists $m \in \mathbf{N}$ such that

$$(t_m \cdot g_m) \circ (t_{m-1} \cdot g_{m-1}) \circ \cdots \circ (t_1 \cdot g_1) = 0.$$

From this it follows that

$$\begin{aligned} \pi(t_m \cdot g_m \circ \cdots \circ t_1 \cdot g_1) &= \pi(t_m \cdot g_m) \circ \cdots \circ \pi(t_1 \cdot g_1) \\ &= (t_m s_m \cdot g'_m) \circ \cdots \circ (t_1 s_1 \cdot g'_1) \\ &= 0 \end{aligned}$$

But, since $\Pi_{j=1}^m t_j s_j$ is an unity in $R_{\mathfrak{p}}$, we get that $g'_m \circ \cdots \circ g'_1 = 0$.

Finally, if we consider a flat envelope $h : M_{\mathfrak{p}} \to F$ of the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ (which exists by Corollary 4) by using Lemmas 6 and 7 we may assume that F is a direct summand of $P_{\mathfrak{p}}$. Then it is easy to see that if $\{k_n\}_{n \in \mathbf{N}}$ is a sequence of endomorphisms of F such that $k_n \circ h = 0$ for each $n \in \mathbf{N}$, there exists $m \in \mathbf{N}$ for which $k_m \circ \cdots \circ k_1 = 0$ and from [2, Prop. 4.2] we obtain that the homomorphism $h^{(\mathbf{N})} : M_{\mathfrak{p}}^{(\mathbf{N})} \to F_{\mathfrak{p}}^{(\mathbf{N})}$ is a flat envelope of the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}^{(\mathbf{N})}$.

For a ring R we will denote by $\mathrm{wD}(R)$ the weak global dimension of R. We can get

THEOREM 9. *Let R be a commutative ring such that* $\mathrm{wD}(R_{\mathfrak{p}}) < \infty$ *for each prime ideal $\mathfrak{p}$ of R. The following conditions are equivalent.*

(i) *R is coherent and* $\mathrm{wD}(R) \leqq 2$.

(ii) *Every R-module has a flat envelope.*

PROOF. (i)$\Rightarrow$(ii) It is [5, Theorem 2.11 (i)$\Rightarrow$(ii)].

(ii)$\Rightarrow$(i) If every R-module has a flat envelope, then R is coherent. Therefore, for each prime ideal $\mathfrak{p}$ of R, $R_{\mathfrak{p}}$ is a coherent ring and as $\mathrm{wD}(R_{\mathfrak{p}}) < \infty$ every principal ideal of $R_{\mathfrak{p}}$ has finite projective dimension and hence $R_{\mathfrak{p}}$ is a domain by [7, Coroll. 5.16]. On the other hand, since every finitely presented $R_{\mathfrak{p}}$-module can be considered as the localization of some finitely presented R-module, we get from Proposition 8 that every countable direct sum of copies of a finitely presented R_p-module M has a flat envelope. Now, by using [2, Corollary to Lemma 5.1], we get that if $M \to F$ is a flat envelope of M, then M and F have the same rank and every diagram of the form

$$\begin{array}{ccc} M & \longrightarrow & F \\ \downarrow & \swarrow & \\ F' & & \end{array}$$

where F' is flat can be completed in a unique way.

Finally, an argument analogous to that used in [5, Theorem 2.11] allows us to obtain the existence of a flat envelope for every R_p-module, and hence from [5, Theorem 2.12] we get that $\mathrm{wD}(R_p) \leqq 2$ so that $\mathrm{wD}(R) = \mathrm{Sup}\{\mathrm{wD}(R_p)\} \leqq 2$ and the proof is complete.

Corollary 10. *Let R be a commutative ring with* $\mathrm{wD}(R) < \infty$. *Then every R-module has a flat envelope if and only if R is coherent and* $\mathrm{wD}(R) \leqq 2$.

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